

OMEGA-LIMIT SETS AND INVARIANT CHAOS IN DIMENSION ONE

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ABSTRACT. Omega-limit sets play an important role in one-dimensional dynamics. During last fifty year at least three definitions of basic set has appeared. Authors often use results with different definition. Here we fill in the gap of missing proof of equivalency of these definitions.

Using results on basic sets we generalize results in paper [P. Oprocha, Invariant scrambled sets and distributional chaos, Dyn. Syst. 24 (2009), no. 1, 31–43.] to the case continuous maps of finite graphs. The Li-Yorke chaos is weaker than positive topological entropy. The equivalency arises when we add condition of invariance to Li-Yorke scrambled set.

In this note we show that for a continuous graph map properties positive topological entropy; horseshoe; invariant Li-Yorke scrambled set; uniform invariant distributional chaotic scrambled set and distributionally chaotic pair are mutually equivalent.

1. INTRODUCTION AND MAIN RESULT

In their famous paper [10] Li and Yorke defined the notion of scrambled set. Their approach of a set containing points which form so-called Li-Yorke pair then became one of the most acceptable definitions of deterministic chaos. In 1986 Smítal ([16]) proved that Li-Yorke chaos is weaker than positive topological entropy, for interval maps.

Since Li and Yorke paper many other definitions of chaos were offered. One of the important generalizations has been provided by Schweizer and Smítal in [17]. Popularity of their distributional chaos comes from the fact that it is equivalent to positive topological entropy in one dimensional cases, contrary to Li-Yorke chaos. In this note we focus on chaos of maps of topological graphs.

An *arc* is any topological space homeomorphic to the compact interval $[0, 1]$. A *graph* is a continuum (a nonempty compact connected metric space) which can be written as the union of finitely many arcs any two of which can intersect only in their endpoints (i.e., it is a one-dimensional compact connected polyhedron). We endow a graph G with metric ϱ of the shortest path. In such setting we consider a continuous map $f: G \rightarrow G$.

For any $x \in G$, the set of accumulation points of the sequence $(f^n(x))_{n=1}^{\infty}$ is called the ω -limit set of x . Let $A \subset G$, by $\text{Orb } A$ we understand the set $\{f^n(x); x \in A \text{ and } n \in \mathbb{N}\}$. We say that f has a *horseshoe* (f is *turbulent*) if there are disjoint arcs U and V such that

$$f(U) \cap f(V) \supset U \cup V.$$

We say that f has a *Li-Yorke pair* if there are points x, y such that

$$\liminf_{n \rightarrow \infty} \varrho(f^n(x), f^n(y)) = 0, \quad \text{and} \quad \limsup_{n \rightarrow \infty} \varrho(f^n(x), f^n(y)) > 0.$$

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Now we proceed with the definition of distributional chaos. For any two points $x, y \in G$, any positive integer n , and any real t we define

$$\xi(x, y, n, t) = \#\{i; 0 \leq i < n \text{ and } \varrho(f^i(x), f^i(y)) < t\}.$$

Now define the *upper distributional* and *lower distributional functions* of the points x and y by formulas

$$F^{xy}(t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \xi(x, y, n, t),$$

$$F_{xy}(t) = \liminf_{n \rightarrow \infty} \frac{1}{n} \xi(x, y, n, t)$$

respectively. The points x and y form a *DC1-pair* for f if

$$F^{xy} \equiv \chi_{(0, \infty)} \text{ and } F_{xy}(\varepsilon) = 0, \text{ for some } \varepsilon > 0,$$

they form a *DC2-pair* if

$$F^{xy} \equiv \chi_{(0, \infty)} \text{ and } F_{xy}(0+) < 1,$$

and finally they form a *DC3-pair* if

$$F^{xy}(t) > F_{xy}(t), \text{ for any } t \text{ in an interval.}$$

Obviously every DC1-pair is a DC2-pair and every DC2-pair is a DC3-pair.

The notion of distributional chaos was introduced in [17], but without naming versions explicitly. In the same paper there was proved that for interval maps all three versions are mutually equivalent and also equivalent to positive topological entropy. These three versions was explicitly named in [18, 1] and discussed their properties in general spaces and particularly in the case of skew-product maps. The relations between the versions of distributional chaos as well as the relations to other important notions as topological entropy, horseshoes and other types of chaos was studied in the case of circle, graphs and dendrites (see [7, 12]). In the case of graph maps all three types are mutually equivalent and therefore we can formulate the last condition of our main theorem for any type of DC-pair.

Let $S \subset G$ be a set containing more than one point. We say that S is a *Li-Yorke scrambled set* if any two different points from S form a Li-Yorke pair. Similarly we define *DC1 (DC2 and DC3) scrambled set*. There are several ways defining f to be chaotic (in Li and Yorke or distributional case) based on the size of its scrambled set — two-point, infinite or uncountable. We refer to [6] for further reading on the size of scrambled sets. In the case of continuous graph maps the existence of a two-point distributional scrambled set is equivalent to the existence of an uncountable one, similarly for Li-Yorke scrambled sets (see [8] for details).

If a scrambled set S satisfies condition $f(S) \subset S$ we call it *invariant*. As it was mentioned, the existence of a Li-Yorke scrambled set is weaker than positive topological entropy. This is no longer true when we add the assumption of invariance of a scrambled set. This was proved in [13] for interval maps and here we generalize the results for the case of graph maps.

A DC1 scrambled set S is called *uniform* if there is an $\varepsilon > 0$ such that $F_{xy}(\varepsilon) = 0$, for any pair of different points x and y from S .

We recall the definition of the topological entropy by Bowen ([4]) and Dinaburg ([5]). Let $\varepsilon > 0$ and n be a positive integer. A set $A \subset G$ is an (n, ε) -*separated set* if for each

$x, y \in A$, $x \neq y$, there is an integer i , $0 \leq i < n$, such that $\varrho(f^i(x), f^i(y)) > \varepsilon$. Let $s_n(\varepsilon)$ denote the maximal cardinality of all (n, ε) -separated sets. The *topological entropy* of f is

$$(1) \quad h(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varepsilon).$$

Let $f: G \rightarrow G$ and $g: K \rightarrow K$ be continuous maps of graphs. A continuous surjection $\varphi: G \rightarrow K$ which is monotone (i.e. $\varphi^{-1}(y)$ is connected for all $y \in K$) is called *semiconjugation* if $\varphi \circ f = g \circ \varphi$. Moreover if there exists an $k \in \mathbb{N}$ such that $\#\varphi^{-1}(x) \leq k$ for each $y \in K$ then we say that φ semiconjugates f and g *almost exactly*.

2. BASIC SETS

In the sixties, A. N. Sharkovsky has systematically studied properties of the ω -limit sets of the continuous maps of the interval (cf., e.g., [14] and [15]). He shown that any such set is contained in a unique maximal ω -limit set and introduced three types of maximal ω -limit set: cycle, first type (lately known as solenoids) and second type (lately known as basic sets).

Definition 1 (S-basic set). *A maximal ω -limit set ω for continuous map of compact interval is called a basic set if ω is infinite and it contains a periodic point.*

Although Sharkovsky's definition was originally formulated for the case of continuous maps of compact interval, it can be used in the same form for graph maps too.

Lately in the eighties A. Blokh widely extend properties of basic sets but using different definition (cf. e.g. [2]).

Definition 2 (B-basic set). *Let I be an n -periodic interval, $\text{Orb } I = M$. Consider a set $B(M, f) = \{x \in M : \text{for any relative neighborhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\text{Orb } U} = M\}$. Set $B(M, f)$ is called a basic set if it is infinite.*

This definition is used, without explicit formulation, in papers [3] for the case of graph maps by taking periodic closed connected subgraph instead of I .

In paper [7] was Sharkovsky's classification of maximal ω -limit sets extended to the case of graph maps. It was necessary to add fourth type of maximal ω -limit sets to cover graph specific cases of ω -limit sets (singular sets). Let ω be a maximal ω -limit set of a continuous map f of graph. Put

$$P_\omega = \bigcap_U \overline{\text{Orb } U}$$

where U is taken over all neighborhood intersecting ω . Note that P_ω is closed and strongly invariant i.e. $f(P_\omega) = P_\omega$. Using cardinality of P_ω and relation ω to the set of periodic points we can divide the set of all maximal ω -limit sets into four classes.

If ω is finite then it is a *cycle*. Now consider ω to be infinite. If P_ω is a nowhere dense set then we call ω a *solenoid*. If P_ω consists of finitely many connected components and ω contains no periodic point then we call ω a *singular set*.

Definition 3 (HM-basic set). *If P_ω consists of finitely many connected components and ω contains a periodic point then ω is called a basic set.*

Use of results on basic sets under different definitions become folklore nowadays. Here we offer proof of equivalency of mentioned definition of basic sets.

Theorem 1. *Definitions of S-basic set, B-basic set and HM-basic set are mutually equivalent.*

Proof. Obviously every HM-basic set is S-basic set. Conversely, let ω is S-basic set, then by must be one of class (in sense of [7]) i.e. cycle, solenoid, singular set and HM-basic set. Since ω is infinite it cannot be cycle. It also cannot be singular set because ω contains a cycle. And if U is a neighborhood of a point of ω then by [17, Theorem 3.7] $\overline{\text{Orb } U}$ covers ω . Therefore ω cannot be solenoid and ω must be HM-basic set.

Let ω is a HM-basic set. Then P_ω is finite union of periodic subgraphs and we put $M = P_\omega$. For each $x \in \omega$ and its neighborhood U we have $\overline{\text{Orb } U} \supset M$. This follows that ω is B-basic set. Now let ω be B -basic set. Again we use fact that ω must be one of mentioned class. It cannot be a cycle since it is infinite and singular set as well because it contains periodic point (see [3, Theorem 4.1]). Finally by [3] ω is maximal and f has positive topological entropy on ω which is impossible on solenoids. Therefore ω is a HM-basic set. \square

Having equivalency of mentioned definition we can formulate the following theorem based on results from [2], [3] and [7].

Theorem 2. *Let f be a continuous graph map and ω its basic set then*

- (i) ω is a perfect set;
- (ii) the system of all basic sets of f is countable;
- (iii) $h(f) \geq (\log 2)/(2n)$, where n is number of periodic portion of ω ;
- (iv) periodic points are dense in ω ;
- (v) $f|_\omega$ is semiconjugated with g almost exactly, where g is a continuous transitive map of a graph;
- (vi) f^n has a horseshoe, for some n ;
- (vii) let P_ω contains no proper periodic subgraph, $U \subset \text{int } P_\omega$ and $J \cap \omega$ be infinite then $f^n(J) \supset U$ for sufficiently large n .

3. INVARIANT CHAOS

The following theorem is the main result of this note.

Theorem 3. *Let f be a continuous graph map then the following conditions are equivalent:*

- (i) topological entropy of f is positive;
- (ii) there exists an $n \in \mathbb{N}$ such that f^n has a horseshoe (is turbulent);
- (iii) there exists an $n \in \mathbb{N}$ such that f^n has an invariant Li-Yorke scrambled set;
- (iv) there exists an $n \in \mathbb{N}$ such that f^n exhibits uniform distributional chaos with an invariant DCI-scrambled set;
- (v) f has a DC pair of any type.

The following lemma is known for interval maps, we generalize it for the case of graph maps.

Lemma 4. *Let f be a continuous graph map. Topological entropy of f is positive if and only if there is an ω -limit set containing a cycle but different from this cycle.*

Proof. Suppose that there is an ω -limit set ω containing a cycle but different from this cycle. By [11] ω is contained in a maximal ω -limit set $\tilde{\omega}$. Set $\tilde{\omega}$ must be infinite (otherwise it would be a cycle) and therefore $\tilde{\omega}$ is a basic set. By Theorem 13 in [7] f has positive entropy.

The converse implication easily follows from Theorem 13 in [7] \square

Lemma 5. *Let f be a continuous graph map. If f has an invariant Li-Yorke scrambled set then f has positive topological entropy.*

Proof. Let x be a point of invariant Li-Yorke scrambled set for f . The condition

$$\liminf_{i \rightarrow \infty} \varrho(f^i(x), f^{i+1}(x)) = 0$$

implies that $\omega_f(x)$ contains a fixed point. On the other hand from the condition

$$\limsup_{i \rightarrow \infty} \varrho(f^i(x), f^{i+1}(x)) > 0$$

we get $\#\omega_f(x) > 1$. We finish the proof by applying the previous lemma. \square

Proof of Theorem 3. The equivalency of (i) and (ii) has been proved in [9].

When f^n has a horseshoe then on a closed invariant subset f^n is conjugate to the full one-sided shift of two symbols. By Theorem 1 in [13] this shift has a uniform invariant DC1-scrambled set. By conjugacy we get scrambled set for f^n with required properties. This proves (ii) implies (iv).

The implication from (iv) to (i) follows from Theorem 13 in [7].

Now apply Lemma 5 to f^n we have that (iii) implies that $h(f^n) > 0$. Then by formula $h(f^n) = nh(f)$ we get (i).

The implication from (iv) to (iii) is trivial.

The equivalency of (v) and (i) has been proved in [12]. \square

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